

# Vorticity Statistics and the Time Scales of Turbulent Strain

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## Abstract

Time scales of turbulent strain activity, denoted as the strain persistence times of first and second order, are obtained from time-dependent expectation values and correlation functions of lagrangian rate-of-strain eigenvalues taken in particularly defined statistical ensembles. Our approach is based on a gaussian closure approximation for single-point vorticity statistics, devised to yield a simple and computationally efficient method for the evaluation of the strain persistence times. We find that softly divergent prefactors correct the usual “ $1/s$ ” time-scale estimate of standard turbulence phenomenology, a fact that turns out to be of fundamental importance in the emergence of turbulent intermittency. The theoretical framework is supported by a straightforward analysis of direct numerical simulation data.

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## I. INTRODUCTION

It is a point of reasonable consensus that further progress in the statistical theory of turbulence has been hampered in great part due to the fact that one of its phenomenological pillars – the Kolmogorov-Richardson cascade – is actually a longstanding open issue. The usual assumption of vortex stretching as the essential mechanism for the local flow of turbulent kinetic energy towards smaller scales has been challenged by the visualization of multiscale vortical structures in real and numerical experiments [1–5], and the related discovery of geometrical statistics phenomena [6, 7]. One may expect that significant advances in the derivation of the statistical properties of turbulence will follow from a deeper understanding of flow instabilities and their role in the production of coherent structures, within more elaborate discussions of the coupled dynamics of vorticity and the rate-of-strain tensor.

A fundamental problem in this context is to determine for how long a given fluid element is, in its lagrangian evolution, coherently compressed or stretched by the underlying strain field. According to common wisdom [8], if  $s$  is some measure of the strain strength, such a “strain persistence time” can be estimated as  $T(s) \sim 1/s$ . However, this expression for  $T(s)$  is in fact problematic, since the constancy of  $sT(s)$  suggests that weak large scale and strong small scale rate-of-strain fluctuations would, respectively, (i) break statistical isotropy at small scales and (ii) have no role in the production of coherent structures, as vortex tubes. Both of these implications are at variance with experimental and numerical observations [9].

Motivated by the above difficulties, our aim in this work is to show that instead of a single time scale  $T(s)$ , the strain activity is more naturally associated to two distinct time scales, which will be denoted as the strain persistence times of first and second order. It turns out that these time scales contain divergent prefactors which multiply the usual  $1/s$  estimate of standard phenomenology, a fact that throws some light on the origin of turbulent intermittency and is perhaps related to the well-reported confinement of strong vorticity fields in small scale coherent structures.

This paper is organized as follows. In the next section we address formal definitions of the strain persistence times and discuss, by means of a straightforward closure scheme, their relation to single-point vorticity statistics. In Sec. III, we verify, taking direct numerical simulation (DNS) data into account, that our analytical framework, devised to hold in prin-

ciple in the small strain domain, incidentally holds for the whole range of strain strengths. In Sec. IV, we comment on our findings and point out directions of further research.

## II. STRAIN PERSISTENCE TIMES

Let  $s_{ij} = (\partial_i v_j + \partial_j v_i)/2$  be the  $(i, j)$ -component of the lagrangian rate-of-strain tensor. Recalling that  $s_{ij}$  is traceless due to incompressibility, call the only positive or the only negative eigenvalue of  $s_{ij}(t)$  by  $\bar{s}(t)$ , a piecewise continuous function of time, as indicated in Fig. 1a.

Independent turbulent flow realizations of  $\bar{s}(t)$  generated, for instance, from some set of random initial conditions at  $t \rightarrow -\infty$  constitute a large functional space  $S$ . Take the ensemble  $\bar{\Lambda}_s \subset S$  of all the profiles  $\bar{s}(t)$  which have  $\bar{s}(0) = s$  for an arbitrarily prescribed eigenvalue  $s$ . Alternatively, we define the related ensemble  $\Lambda_s$  of compactly supported functions  $s(t)$  which are identified to  $\bar{s}(t) \in \bar{\Lambda}_s$  in the largest neighborhood of  $t = 0$  where  $\bar{s}(t)$  is continuous. The functions  $s(t)$  vanish out of these neighborhoods. See the sketches in Fig. 1b.

In a more formal way,  $\Lambda_s$  is given as the ensemble of functions  $s(t)$  obtained from the one-to-one mapping

$$\begin{aligned} \bar{\Lambda}_s &\mapsto \Lambda_s \\ \bar{s}(t) &\rightarrow s(t) , \end{aligned} \tag{2.1}$$

given, for positive  $t$ , by

$$s(t) = \begin{cases} \bar{s}(t) & \text{if } \forall t' \in [0, t] , \bar{s}(t')/s > 0 , \\ 0 & \text{if } \exists t' \in [0, t] \mid \bar{s}(t')/s < 0 , \end{cases} \tag{2.2}$$

while for negative  $t$ , the time interval  $[0, t]$  is replaced, in Eq. (2.2), by  $[t, 0]$ .

The importance of the ensemble  $\bar{\Lambda}_s$  relies on the fact that their elements, i.e., the time-dependent strain eigenvalues  $\bar{s}(t)$ , have all the same postulated “strain strength”  $s \equiv \bar{s}(0)$ . Our task, therefore, is to investigate their characteristic time scales and to understand how they depend on  $s$ . Furthermore, the rationale for the introduction of the additional ensemble  $\Lambda_s$ , besides  $\bar{\Lambda}_s$ , is that it leads to expectation values which are less contaminated by errors than the ones computed through  $\bar{\Lambda}_s$ , while retaining the same (or approximately the same) relevant time scales of strain activity.

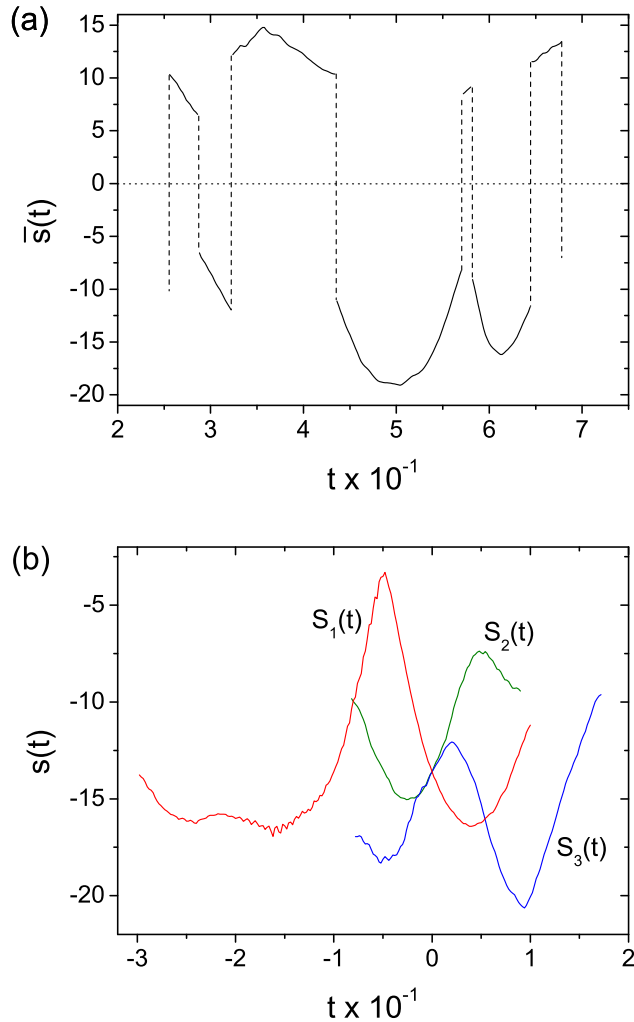


FIG. 1: Figs. (a) and (b) are produced from the direct numerical simulation (DNS) data which is addressed in Sec. III. In (a) we show a typical profile of the piecewise continuous function  $\bar{s}(t)$ . In (b) the compactly supported configurations  $s_1(t)$ ,  $s_2(t)$  and  $s_3(t)$  belong all to the ensemble  $\Lambda_s$ , with  $s = -13.6 \pm 0.1$ , since  $s_1(0) = s_2(0) = s_3(0) = s$ . The total time span of the DNS is  $\Delta t = 2048 \times 10^{-3}$  in arbitrary time units.

We emphasize that the just given definitions of the ensembles  $\bar{\Lambda}_s$  and  $\Lambda_s$  are strong idealizations of dubious utility if the interest is to compute statistical averages out of experimental or numerical data. The essential difficulty here is that one should work, in principle, with a large functional space of turbulent flow realizations. However, by evoking ergodicity [10], a statistically equivalent ensemble  $\Lambda_s$  can be introduced for practical purposes as it follows:

(i) Taking a single three-dimensional turbulent flow realization (e.g., the one obtained in a direct numerical simulation) recorded in a time interval of length  $T$ , pick up a number  $N$  of (hopefully weakly correlated) lagrangian trajectories;

(ii) An alternative set of time parametrizations of the lagrangian trajectories is implemented by conventionally setting time  $t = 0$  at  $M$  equally spaced time instants along the dynamical evolution;

(iii) Define the ensemble  $\bar{\Lambda}_s$  of  $N \times M$  strain eigenvalue profiles  $\bar{s}(t)$  obtained from the lagrangian trajectories introduced in (i) and time-parametrized according to (ii);

(iv) Working with some assigned uncertainty  $\delta s$  in the definition of  $s$ , introduce the ensemble  $\Lambda_s$  of functions  $s(t)$  derived from the profiles defined in (iii).

In the course of analysis, we will be particularly interested in the following time integrations over first and second order expectation values taken in  $\Lambda_s$ ,

$$I_1(s) = \int_{-\infty}^0 dt \langle s(t) \rangle_{\Lambda_s} , \quad (2.3)$$

$$I_2(s) = \int_{-\infty}^0 dt \int_{-\infty}^0 dt' \langle \delta s(t) \delta s(t') \rangle_{\Lambda_s} , \quad (2.4)$$

where

$$\delta s(t) \equiv s(t) - \langle s(t) \rangle_{\Lambda_s} . \quad (2.5)$$

Considering, for convenience, positive and negative  $s$  as separate cases, strain persistence times of first and second order,  $T_1^\pm(s)$  and  $T_2^\pm(s)$ , respectively, can be defined from (2.3) and (2.4), as

$$I_1(s) = \begin{cases} s T_1^+(s) & \text{for } s > 0 , \\ s T_1^-(s) & \text{for } s < 0 , \end{cases} \quad (2.6)$$

and

$$I_2(s) = \begin{cases} [s T_2^+(s)]^2 & \text{for } s > 0 , \\ [s T_2^-(s)]^2 & \text{for } s < 0 . \end{cases} \quad (2.7)$$

The central idea underlying our discussion is that analytical expressions for  $T_1^\pm(s)$  and  $T_2^\pm(s)$  can be derived from a statistical treatment of vorticity fluctuations, which are governed by the lagrangian evolution equation [11],

$$\frac{d\omega_i}{dt} = s_{ij}(t)\omega_j + \epsilon_{ijk}\partial_j f_k + \nu\partial^2\omega_i , \quad (2.8)$$

where  $f_i$  is the external force and  $\nu$  is the kinematical viscosity. Let  $\hat{n}(t)$  be the unit vector defined along the principal direction associated to the only positive or only negative eigenvalue  $\bar{s}(t)$  of the lagrangian rate-of-strain tensor (the twofold orientation ambiguity of  $\hat{n}(t)$  is arbitrarily resolved). One gets, after some simple algebra on Eq. (2.8),

$$\frac{d\omega^2}{dt} = 2\bar{s}(t)\omega^2 + 2\omega\vec{\omega} \cdot \frac{d\hat{n}}{dt} + \epsilon_{ijk}\omega_i\partial_j f_k + \nu\omega_i\partial^2\omega_i, \quad (2.9)$$

where  $\omega(t) \equiv \vec{\omega}(t) \cdot \hat{n}(t)$ . It is interesting to take the expectation value of Eq. (2.9) conditioned to a given time-dependent profile  $\bar{s}(t) \in \bar{\Lambda}_s$ . In other words, we change our focus to the alternative evolution equation

$$\frac{d\langle\omega^2\rangle_{[s]}}{dt} = 2\bar{s}(t)\langle\omega^2\rangle_{[s]} + 2\langle\omega\vec{\omega} \cdot \frac{d\hat{n}}{dt}\rangle_{[s]} + \epsilon_{ijk}\langle\omega_i\partial_j f_k\rangle_{[s]} + \nu\langle\omega_i\partial^2\omega_i\rangle_{[s]}, \quad (2.10)$$

where  $\langle(\dots)\rangle_{[s]}$  is the self-evident notation for the procedure of conditional averaging. It is clear that Eq. (2.10) is not closed. The second term on its right-hand-side, for instance, is related in an intricate way, through  $d\hat{n}/dt$ , to the velocity gradient tensor and the pressure hessian. Neglecting small alignment effects [6], we take, as a first approximation, that  $\vec{\omega}(t)$  and  $\hat{n}(t)$  are completely uncorrelated in the ensemble of flow realizations which share the same arbitrary profile  $\bar{s}(t)$ . Resorting furthermore to isotropy, it follows, thus, that

$$\langle\omega\vec{\omega} \cdot \frac{d\hat{n}}{dt}\rangle_{[s]} = \langle\omega_i\omega_j\rangle_{[s]}\langle\hat{n} \cdot \hat{e}_i \frac{d}{dt}(\hat{n} \cdot \hat{e}_j)\rangle_{[s]} = \frac{1}{3}\langle\vec{\omega}^2\rangle_{[s]}\langle\hat{n} \cdot \frac{d\hat{n}}{dt}\rangle_{[s]} = 0. \quad (2.11)$$

The last two terms in Eq. (2.10) are just the rates of enstrophy injection and dissipation, so that their combined contribution is assumed to vanish. Taking Eq. (2.11) into account, we are led, after a straightforward integration of Eq. (2.10), to

$$\langle\omega^2(t)\rangle_{[s]} = \langle\omega^2(T)\rangle_{[s]} \exp\left[2\int_T^t dt' \bar{s}(t')\right]. \quad (2.12)$$

Eq. (2.12) is now averaged over the configurations  $\bar{s}(t) \in \bar{\Lambda}_s$ . This can be rethorically expressed in path-integral language [12] as

$$\langle\omega^2(0)\rangle_{\bar{\Lambda}_s} = \int_{\bar{\Lambda}_s} D[\bar{s}(t)]\rho[\bar{s}(t)]\langle\omega^2(T)\rangle_{[s]} \exp\left[2\int_T^0 dt' \bar{s}(t')\right], \quad (2.13)$$

where  $D[\bar{s}(t)]\rho[\bar{s}(t)]$  is the probability measure defined on  $\bar{\Lambda}_s$ . Considering  $T$  to be large negative enough (typically much larger, in absolute value, than the large eddy turnover time), the squared vorticity expectation value which appears in the integrand of (2.13)

becomes, supposedly, independent on the elements of  $\bar{\Lambda}_s$  and on the particular time  $T$ . We get, therefore,

$$\langle \omega^2(0) \rangle_{\bar{\Lambda}_s} = c^2 \lim_{T \rightarrow -\infty} \int_{\bar{\Lambda}_s} D[\bar{s}(t)] \rho[\bar{s}(t)] \exp \left[ 2 \int_T^0 dt' \bar{s}(t') \right] , \quad (2.14)$$

where  $c^2 \equiv \langle \omega^2(T) \rangle_{\bar{\Lambda}_s}$ .

The cumulant expansion method [13] can, at this point, be straightforwardly applied to Eq. (2.14) to yield, in a gaussian closure approximation (i.e., up to second order in the strain fluctuations),

$$\langle \omega^2(0) \rangle_{\bar{\Lambda}_s} = c^2 \exp \left[ 2 \int_{-\infty}^0 dt \langle \bar{s}(t) \rangle_{\bar{\Lambda}_s} + 2 \int_{-\infty}^0 dt \int_{-\infty}^0 dt' \langle \delta \bar{s}(t) \delta \bar{s}(t') \rangle_{\bar{\Lambda}_s} \right] , \quad (2.15)$$

Noticing that the functions  $\bar{s}(t) \in \bar{\Lambda}_s$  are essentially extensions of the corresponding  $s(t) \in \Lambda_s$  beyond strain persistence time scales, we can replace, as an approximation,  $\bar{s}(t)$  by  $s(t)$  and  $\bar{\Lambda}_s$  by  $\Lambda_s$  in the right hand side of Eq. (2.15). We obtain, thus, from Eqs. (2.3) to (2.7) and (2.15),

$$\langle \omega^2(0) \rangle_{\bar{\Lambda}_s} = \begin{cases} c^2 \exp \left\{ 2sT_1^+(s) + 2[sT_2^+(s)]^2 \right\} & \text{for } s > 0 , \\ c^2 \exp \left\{ 2sT_1^-(s) + 2[sT_2^-(s)]^2 \right\} & \text{for } s < 0 . \end{cases} \quad (2.16)$$

It is convenient to write, up to first order in  $s$ , the standard deviation of  $\omega(0)$  in the ensemble  $\bar{\Lambda}_s$  as

$$\sigma_\omega(s) \equiv \sqrt{\langle \omega^2(0) \rangle_{\bar{\Lambda}_s}} = \begin{cases} a(1 + b_+ s) & \text{for } s > 0 , \\ a(1 - b_- s) & \text{for } s < 0 , \end{cases} \quad (2.17)$$

where  $a > 0$ ,  $b_+$  and  $b_-$  are arbitrary coefficients and we have used that  $\vec{\omega}$  and  $\hat{n}$  are independent random variables in  $\bar{\Lambda}_s$ , so that  $\langle \omega(0) \rangle_{\bar{\Lambda}_s} = 0$ . As reported in the next section, we have found convincing numerical evidence that there are no higher order corrections to Eq. (2.17). This is a remarkable fact, still in need of understanding.

As stated (in rephrased form) in the introductory section, we do not expect vanishing strain to have any effect on the statistics of vorticity. Requiring, therefore, that

$$\lim_{s \rightarrow 0} [sT_1^\pm(s)] = \lim_{s \rightarrow 0} [sT_2^\pm(s)] = 0 , \quad (2.18)$$

it follows from Eqs. (2.16) and (2.17) that  $c = a$ .

Taking into account, now, the time-reversal symmetry of the fluid dynamic equations in the absence of forcing and dissipation terms, a meaningful approximation for the description of inertial range processes, we assume that  $T_1^+(s) \propto T_1^-(-s)$  and  $T_2^+(s) \propto T_2^-(-s)$ . We point

out that this argument is not inconsistent at all with the dissipation anomaly postulated by the “zeroth law” of turbulence, that is, the fact that energy dissipation rate per unit volume is finite in the inviscid limit  $\nu \rightarrow 0$  [11, 14–17]. The situation here is analogous to the issue on the coexistence of the second law of thermodynamics with microscopic reversibility in the statistical mechanics context. Introducing a pair of even functions of  $s$ ,  $F_1(s)$  and  $F_2(s)$ , and proportionality constants  $g$  and  $g'$ , we may write, thus, without loss of further generality, that

$$T_1^+(s) = \frac{1}{s} \ln F_1(s) , \quad T_1^-(s) = -\frac{g}{s} \ln F_1(s) , \quad (2.19)$$

$$[T_2^+(s)]^2 = \frac{1}{s^2} \ln F_2(s) , \quad [T_2^-(s)]^2 = \frac{g'}{s^2} \ln F_2(s) . \quad (2.20)$$

Considering, below, that  $s > 0$ , note that

$$\frac{I_1(-s)}{I_1(s)} = \frac{\int_{-\infty}^0 dt \langle s(t) \rangle_{\Lambda_{-s}}}{\int_{-\infty}^0 dt \langle s(t) \rangle_{\Lambda_s}} = -\frac{T_1^-(-s)}{T_1^+(s)} = -g \quad (2.21)$$

and

$$\frac{I_2(-s)}{I_2(s)} = \frac{\int_{-\infty}^0 dt \int_{-\infty}^0 dt' \langle \delta s(t) \delta s(t') \rangle_{\Lambda_{-s}}}{\int_{-\infty}^0 dt \int_{-\infty}^0 dt' \langle \delta s(t) \delta s(t') \rangle_{\Lambda_s}} = \left[ \frac{T_2^-(-s)}{T_2^+(s)} \right]^2 = g' . \quad (2.22)$$

From Eqs. (2.16), (2.19) and (2.20) and the even parity of  $F_1(s)$  and  $F_2(s)$ , it turns out that Eq. (2.17) holds if and only if

$$F_1(s) \cdot F_2(s) = 1 + b_+ |s| , \quad (2.23)$$

$$\frac{[F_2(s)]^{g'}}{[F_1(s)]^g} = 1 + b_- |s| . \quad (2.24)$$

Substituting the solutions of Eqs. (2.23) and (2.24) for  $F_1(s)$  and  $F_2(s)$  into Eqs. (2.19) and (2.20), we find

$$T_1^+(s) = \frac{1}{g + g'} T_1(s) , \quad T_1^-(s) = \frac{g}{g + g'} T_1(s) , \quad (2.25)$$

$$T_2^+(s) = \sqrt{\frac{1}{g + g'}} T_2(s) , \quad T_2^-(s) = \sqrt{\frac{g'}{g + g'}} T_2(s) , \quad (2.26)$$

where

$$T_1(s) = \frac{1}{|s|} \ln \left[ \frac{(1 + b_+ |s|)^{g'}}{1 + b_- |s|} \right] , \quad (2.27)$$

and

$$T_2(s) = \frac{1}{|s|} \sqrt{\ln [(1 + b_+ |s|)^g (1 + b_- |s|)]} . \quad (2.28)$$



It is interesting to remark that once  $T_1^\pm(s)$  is positive definite, it is necessary to have, according to (2.27),  $g' \geq 1$  and  $g'b_+ > b_-$  [18]. This is a simple and well-defined prediction from the present formalism.

In the weak strain regime, one expects that the strain persistence times saturate to the large eddy turnover time  $T_0 \equiv \sqrt[3]{L^2/\epsilon}$ , where  $L$  and  $\epsilon$  are, respectively, the typical large length scale and the energy dissipation rate parameters of the turbulent flow. We get, thus, from Eqs. (2.25) to (2.28),

$$\lim_{s \rightarrow 0} T_1^\pm(s) \propto g'b_+ - b_- \propto T_0, \quad (2.29)$$

$$\lim_{s \rightarrow 0} T_2^\pm(s) \propto gb_+ + b_- \propto T_0. \quad (2.30)$$

Both parameters  $b_+$  and  $b_-$  are, therefore, likely to be proportional to the large eddy turnover time  $T_0$ . An interesting problem, not touched here, is to find the Reynolds number dependence, if any, of the dimensionless parameters  $g$ ,  $g'$ ,  $b_+/T_0$ , and  $b_-/T_0$ .

### III. ANALYSIS OF DNS DATA

We have computed statistical averages of fluid dynamic observables with the help of the direct numerical simulation (DNS) database available from the turbulence research group at Johns Hopkins University [19]. An homogeneous and isotropic turbulent flow with Taylor-based Reynolds number  $R_\lambda \approx 433$  is simulated in a periodic cube of linear dimension  $L = 2\pi$  modeled as a grid of  $1024^3$  lattice points. Viscosity and time step parameters are, respectively,  $\nu = 1.85 \times 10^{-4}$  and  $\Delta t = 2 \times 10^{-4}$  (the complete simulation record corresponds to around one large eddy turnover time,  $T_0 = 2^{10} \times 10 \times \Delta t$ ). Further simulation details can be found in Refs. [20, 21].

Statistical samples were produced in two different ways, according to the particular expectation values we were interested to evaluate:

(i) In order to compute  $\sigma_\omega(s)$ , vorticity vectors and rate-of-strain tensors were defined from the velocity gradients taken at grid points  $2\pi(i, 8j, 8k)/1024$  where  $0 \leq i < 1024$  and  $0 \leq j, k < 128$  are integer numbers, for  $10^3$  frames of equally time-spaced flow configurations. Bins of variable sizes were considered for the sets of positive and negative rate-of-strain eigenvalues  $s$ . While essentially conventional, our particular bin size choice,  $\delta s = 0.5$ , proved to yield robust results at economical computational costs.

(ii) For the evaluation of  $I_1(s)$  and  $I_2(s)$  we have considered  $10^3$  lagrangian trajectories, each one consisting of  $2^{10}$  time steps (which are separated in time by  $2 \times 10^{-3}$  arbitrary time units). The initial points of the lagrangian trajectories are given by  $2\pi(i, j, k)/10$ , with  $0 \leq i, j, k \leq 10$ , and have their time evolution determined through a fourth-order Runge-Kutta algorithm. The ensemble  $\Lambda_s$ , with uncertainty  $\delta s = 0.1$ , was generated through the procedure previously discussed in Sec. II.

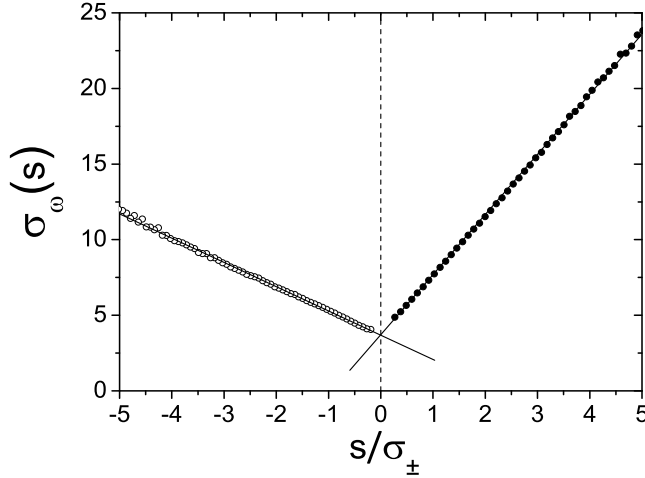


FIG. 2:  $\sigma_\omega(s) = \sqrt{\langle \omega^2(0) \rangle_{\Lambda_s}}$ , i.e., the standard deviation of the projected lagrangian vorticity  $\omega(0)$ , is plotted as a function of standardized rate-of-strain eigenvalues  $s/\sigma_\pm$ . The standard deviations of the positive and negative rate-of-strain eigenvalues  $s$  are, respectively,  $\sigma_+ = 4.63$  and  $\sigma_- = 6.52$ .

In both of the above cases (i) and (ii), the eigenvalues and principal directions of the rate-of-strain tensor were computed through an efficient hybrid algorithm which combines direct analytical evaluation and the so-called QL algorithm [22].

As it is clear from Fig. 2, the conditional expectation value  $\sigma_\omega(s)$  is precisely described by Eq. (2.17), with no additional corrections. The slope parameters  $b_+$  and  $b_-$  associated to the right and left branches of  $\sigma_\omega(s)$ , respectively, are  $b_+ = a^{-1}d\sigma_\omega(s)/ds|_{s>0} = 0.26$  and  $b_- = a^{-1}d\sigma_\omega(s)/ds|_{s<0} = 0.08$ .

In Fig. 3, we show that the expectation value ratios  $I_1(-s)/I_1(s)$  and  $I_2(-s)/I_2(s)$  are approximately constant for an extended range of  $s$ -values, in agreement with Eqs. (2.21) and (2.22). The measured values of  $b_+$ ,  $b_-$ ,  $g = |I_1(-s)/I_1(s)|$  and  $g' = I_2(-s)/I_2(s)$  are then substituted in the logarithmic corrections written down in Eqs. (2.27) and (2.28), which are strikingly confirmed from the plots shown in Fig. 4.

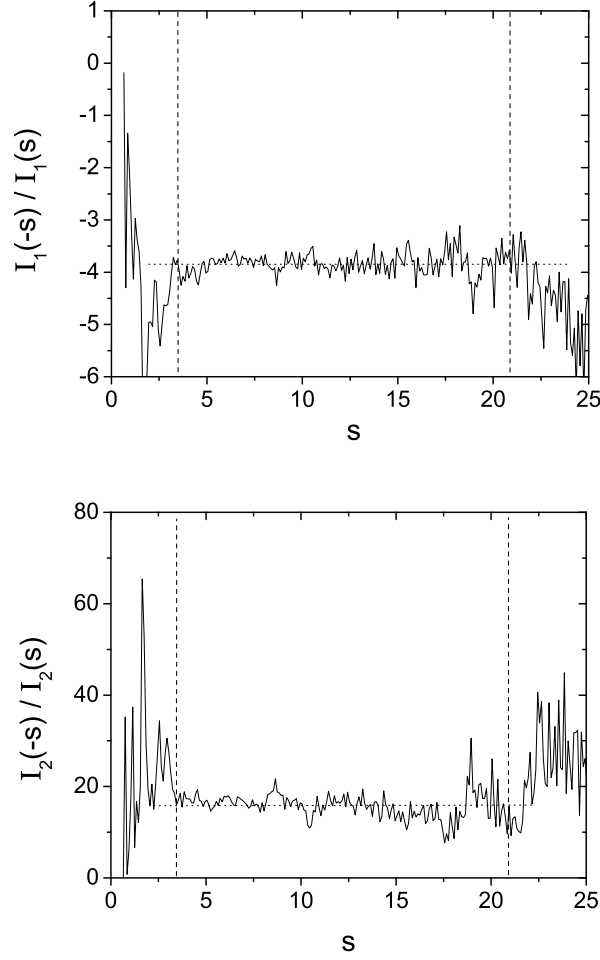


FIG. 3: The horizontal dotted lines indicate the mean values  $-3.84$  ( $\equiv -g$ ) and  $15.81$  ( $\equiv g'$ ) of the ratios  $I_1(-s)/I_1(s)$  and  $I_2(-s)/I_2(s)$ , which are, in fact, approximately constant in the range  $3.4 \leq s \leq 21.0$ , the region between vertical dashed lines, where the ensembles  $\Lambda_s$  are large enough for the evaluation of reasonable statistical averages.

The absolute ratio  $|I_1(-s)/I_1(s)| = T_1(-s)/T_1(s) \simeq 3.84$  is actually expected to be a number larger than unity, since, as it is well-known,  $s$  is most of the time negative, in other words,  $T_1(-s) > T_1(s)$ . We have in fact verified that the domain of negative  $s$  in physical space constitutes around 75% of the total fluid volume, which amounts to say that the intermediate rate-of-strain eigenvalue is positively skewed [6, 23].

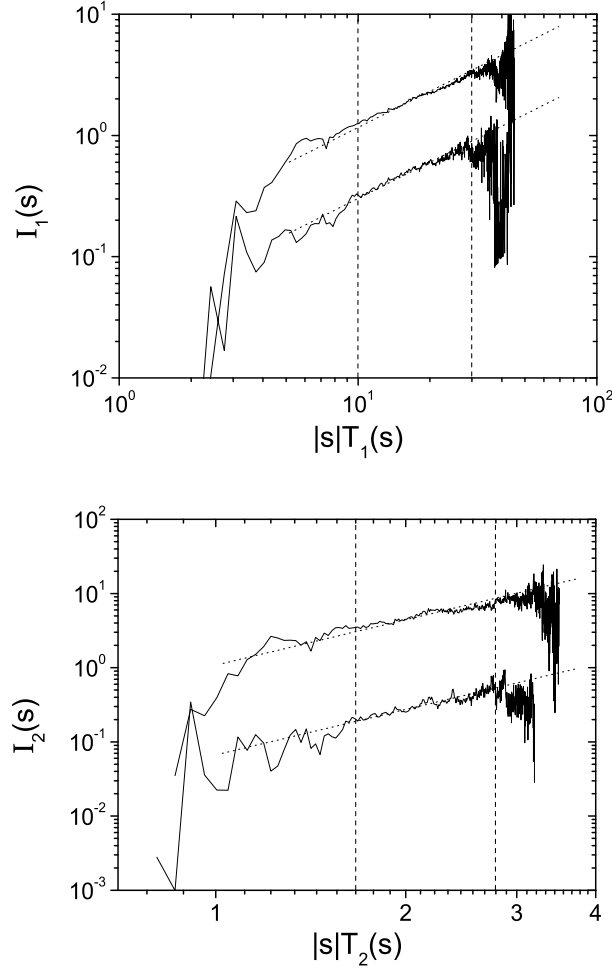


FIG. 4: Plots which provide support for the validity of Eqs. (2.6) and (2.7) with the use of the strain persistence times  $T_1^\pm(s)$  and  $T_2^\pm(s)$ , as defined from Eqs. (2.25)-(2.28). The dotted lines in the plots for  $I_1(s)$  have unit slope, while the ones in the plots for  $I_2(s)$  have slope 2. The upper and lower plots in each pair of plots correspond to negative and positive  $s$ , respectively. These results strongly support the theoretical arguments put forward in Sec. II. The vertical dashed lines indicate the range  $3.4 \leq s \leq 21.0$  (see Fig. 3).

We note, furthermore, that the estimate  $g' \simeq 15.81$  is compatible with the relation  $g'b_+ > b_-$ , predicted at the end of Sec. II.

## IV. CONCLUSIONS

We have devised, within an appropriate statistical framework, a systematic procedure for the evaluation of the time scales associated to turbulent lagrangian strain activity. Two time scales, the strain persistence times  $T_1(s)$  and  $T_2(s)$ , given in Eqs. (2.27) and (2.28), are necessary to accurately reproduce  $\sigma_\omega(s) = \sqrt{\langle \omega^2(0) \rangle_{\Lambda_s}}$ , as determined from DNS data. The strain persistence times  $T_1(s)$  and  $T_2(s)$  are introduced as first and second order contributions of a gaussian closure approximation.

While  $|s|T_1(s)$  and  $|s|T_2(s)$  vanish at  $s = 0$ , they are slowly divergent at asymptotically large  $|s|$ , a fact that happens to be a crucial ingredient in the derivation of the linear profiles of  $\sigma_\omega(s)$ , clearly indicated in Fig. 2. The divergences of  $|s|T_1(s)$  and  $|s|T_2(s)$  as  $|s| \rightarrow \infty$  are actually welcome, since from a phenomenological point of view they imply that the coarse grained enstrophy density should grow at smaller scales, where strain fluctuations are stronger. Also, it is tempting to conjecture that the behavior of the strain persistence times at large  $s$  is related to phenomena like the transient formation of unstable vortex “pancakes”, which tend to roll-up and eventually become vortex tubes, one of the main suspected physical mechanisms of turbulent intermittency.

It is intriguing that the gaussian closure we have implemented here works so well, which seems to suggest that the cumulant expansion captures the essential physics of lagrangian vorticity fluctuations. It is possible that the second-order truncation in the cumulant expansion (2.15) is actually a fine approximation to the full non-perturbative result, thanks to the specific definition of the statistical ensembles  $\Lambda_s$ , which would provide a partition of the whole functional space of strain profiles  $s(t)$  into subspaces of gaussian stochastic processes.

We highlight that from a purely theoretical perspective, no considerations have been advanced to establish the form of the strain persistence times beyond the first order in  $s$ . However, we have found that the empirical evaluation of  $\sigma_\omega(s)$  does not bring any further non-linear corrections into scene, a result which is far from trivial (note that statistical isotropy just implies that  $\langle \vec{\omega}^2 \rangle = 2\langle s_{ij}^2 \rangle$ , which looks like a necessary but by no means a sufficient condition for the specific observed profile of  $\sigma_\omega(s)$ ).

Besides further improvements in the statistical evaluations carried out so far, a natural direction of research is to check to what extent the premises and results proposed in this work can match the phenomenology implied by promising effective lagrangian simulations

of the velocity gradient tensor, as, for instance, the ones derived within the Recent Fluid Deformation Closure model [24, 25].

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